

Non-separating Cycles and Discrete Jordan Curves

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We study induced non-separating cycles in 2-connected and 3-connected graphs. As consequences, two conjectures of Vince and Little on discrete Jordan curves are solved, in particular, they are true for simple 2-connected graphs with minimum degree at least six (this is best possible). We also disprove a conjecture of Thomassen and Toft about the structure of 2-connected graphs in which all induced cycles are separating. © 1992 Academic Press, Inc.

1. INTRODUCTION

We use the terminology of [3]. If not specified, we deal with multigraphs (multiple edges are allowed, but no loops). Let G be a graph. Then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G , respectively. For $S \subset V(G)$ or $S \subset E(G)$, $G[S]$ is the subgraph of G induced by S . If no confusion arises, we usually do not distinguish between a vertex set (or an edge set) and its induced subgraph. Let x be a vertex of G . Then, $N_G(x)$ is the neighborhood of x in G and $d_G(x)$ is the degree of x in G . For two graphs G and H , we use $G \cap H$ to denote the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$.

A cycle is a connected graph in which every vertex is of degree 2. A cycle γ in a graph G is *non-separating* if $G \setminus V(\gamma)$ has the same number of components as G , otherwise γ is called *separating*. A cycle γ is a *discrete Jordan curve* if there exist connected proper subgraphs I and O of G such that $I \cap O = \gamma$ and $I \cup O = H$, where $I \neq \gamma$, $O \neq \gamma$, and H is a connected component of G . Obviously, this is a natural discrete version of a Jordan curve on the sphere.

There is a natural relation between discrete Jordan curves and separating cycles: a separating cycle is also a discrete Jordan curve, and a non-separating cycle is a discrete Jordan curve if and only if it has at least two chords if it is a Hamilton cycle or at least one chord otherwise. Thus, an induced non-separating cycle is not a discrete Jordan curve.

A *cycle double cover* (or simply a CDC) of a graph is a family of cycles such that every edge of the graph appears in exactly two cycles of the family. It is conjectured in [8] that every 2-connected graph has a CDC. While this conjecture is still open, several classes of graphs have been shown to have CDCs, for example, 4-edge-connected graphs [5]. Recently, Lai, Yu, and Zhang [6] have shown that *if a cubic graph on $n \geq 6$ vertices has a CDC, then it has a CDC with at most $n/2$ cycles*. In particular, if a cubic graph on $n \geq 6$ vertices has no subdivision of the Petersen graph, then it has a CDC with at most $n/2$ cycles (see [1]). This result can be used to construct many examples (see Sections 2 and 3).

In Section 2, we discuss two conjectures of Vince and Little [12] on the existence of discrete Jordan curves outside a given CDC. We show that they are not true in general, but true for 3-connected simple graphs by using a strong result of Thomassen [9]. Then in Section 3, we show that for any non-planar simple 3-connected graph there exist three induced non-separating cycles having an edge in common and each missing at least one of two specified vertices. Using this, we show that the two conjectures of Vince and Little are true for simple 2-connected graphs with minimum degree at least six. However, simple counter-examples with minimum degree 2, 3, 4, or 5 are also given. Finally, in Section 4, using the result of Section 3, we give a different proof of a theorem of Thomassen and Toft [10], and we also disprove a conjecture in [10].

2. DISCRETE JORDAN CURVES

A CDC C of G is said to be *reducible* if C has a subfamily F which is a CDC of a proper subgraph of G , otherwise C is *irreducible*. An *irreducible component* of G is a maximal subgraph H of G such that C has a subfamily which is an irreducible CDC of H .

For a set S , we use $|S|$ to denote its cardinality. However, for a CDC C of a graph, it is understood that $|C|$ denotes the number of cycles in C . Following [12], the Euler characteristic of G with respect to a CDC C is $\chi(G, C) = |V(G)| - |E(G)| + |C|$. Note that if C consists of all facial cycles of an embedding of G (on any surface), $\chi(G, C)$ is the usual definition.

It is well known that if $\omega(G)$ is the number of components of G , then the dimension of the cycle space of G is $|E(G)| - |V(G)| + \omega(G)$. Since a CDC C of G consists of cycles of G , the members of C generate a subspace of the cycle space, and we use $\dim(C)$ to denote the dimension of the subspace spanned by C . Let $k(G, C) = |C| - \dim(C)$. It is shown in [12] that $k(G, C)$ is the number of irreducible components of G with respect to C . The following two results are in [12].

THEOREM 2.1. $\chi(G, C) \leq k(G, C) + \omega(G)$, with equality if and only if C spans the cycle space of G .

THEOREM 2.2. If $\chi(G, C) = k(G, C) + \omega(G)$, then every cycle of G not in C is a discrete Jordan curve.

As pointed out by Vince and Little, Theorem 2.2 has a special case for sphere: G is a 2-connected plane graph, C consists of all facial cycles, $k(G, C) = \omega(G) = 1$, and $\chi(G, C) = 2$.

The proof for Theorem 2.2 in [12] is based on Theorem 2.1: since C spans the cycle space of G , for every cycle γ of G there is a subfamily F of C such that $\gamma = \sum_{D \in F} D = \sum_{D \in C \setminus F} D$. Hence, one naturally expects that Theorem 2.2 may be extended to every cycle in the subspace spanned by C by dropping the condition that C spans the cycle space of G . However, this is not the case as shown by the following examples.

The graphs G in Fig. 1 are obtained from two disjoint cycles $\gamma = u_1 u_2 \cdots u_n u_1$ and $v_1 v_2 \cdots v_n v_1$ by joining edges $u_i v_i$, $i = 1, \dots, n$, and by adding a vertex w which is joined to each vertex of these two cycles. Let $C = \{wu_i u_{i+1} w, wv_i v_{i+1} w, u_i u_{i+1} v_i v_{i+1} u_i; i = 1, \dots, n\}$ where the sum in subscripts is modulo n . Clearly, C is an irreducible CDC of G , $\chi(G, C) = (2n + 1) - 5n + 3n = 1 < 2 = k(G, C) + \omega(G)$, and γ is in the subspace spanned by C . But γ is not a discrete Jordan curve (since γ is induced and non-separating).

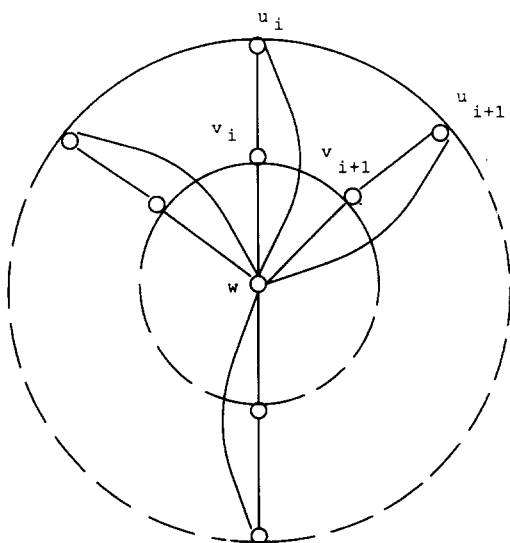


FIGURE 1

Our concern is the converse of Theorem 2.2. In general, the converse is not true when C is reducible (an easy example can be found in [12]). However, Vince and Little [12] made the following two conjectures. Note that if G has an irreducible CDC C , then G is 2-connected and $\chi(G, C) \leq 2$.

Conjecture 1. If $\chi(G, C) \neq 2$ for an irreducible CDC C of G , then there exists a cycle not in C that is not a discrete Jordan curve.

Conjecture 2. If $\chi(G, C) \neq 2$ for an irreducible CDC C of G , then there exists a non-separating cycle not in C .

It is also pointed out in [12] that if the non-separating cycle in Conjecture 2 is also required to be induced, then it is a desired cycle of Conjecture 1. This is clear from the relation between non-separating cycles and discrete Jordan curves discussed in Section 1. However, for simple 3-connected graphs, an even stronger result can be derived from the following result due to Thomassen [9]. (A slightly weaker version of Thomassen's result was originally obtained by Tutte [11]; however, Thomassen's proof is simpler.)

THEOREM 2.3. *If G is a simple 3-connected graph, then every edge of G lies on two induced non-separating cycles with no other common vertex, and all induced non-separating cycles of G span the cycle space of G . Moreover, G is planar if and only if every edge is contained in exactly two induced non-separating cycles, in this case the induced non-separating cycles are precisely the facial cycles of G .*

COROLLARY 2.4. *Let C be a CDC of a simple 3-connected graph G . Then G has an induced non-separating cycle not in C with the only exception that G is planar and C consists of all facial cycles.*

COROLLARY 2.5. *Conjectures 1 and 2 are true for simple 3-connected graphs.*

On the other hand, for simple graphs which are not 3-connected, we have the following counter-examples to both conjectures.

The graphs G in Fig. 2 are obtained from two disjoint cycles γ_1, γ_2 of length ≥ 4 and two other vertices x and y by joining both x and y to two non-consecutive vertices s, t of γ_1 and to two non-consecutive vertices u, v of γ_2 , and by adding the edges st and uv . In the clockwise order, let the path in γ_1 from s to t be P_1 and the one from t to s be P_2 , and let the path in γ_2 from u to v be Q_1 and the one from v to u be Q_2 . It is clear that any cycle of G except γ_1 and γ_2 is separating. Therefore, G has only two non-separating cycles: γ_1, γ_2 . We now extend $\{\gamma_1, \gamma_2\}$ to an irreducible CDC of G as follows: $C = \{\gamma_1, \gamma_2, xsP_1yuQ_1x, xtP_2yvQ_2x, xstx, ysty, xuvx, yuvy\}$.

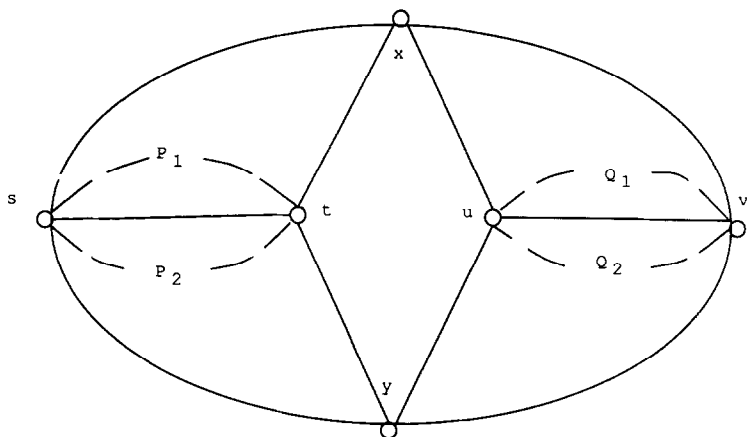


FIGURE 2

An easy calculation shows that $\chi(G, C) = 0 < 2 = k(G, C) + w(G)$. But any cycle not in C is a separating cycle (and so, is a discrete Jordan curve).

Note that the above graphs have minimum degree 2. In the next section, we will show that there are simple counter-examples (to both conjectures) of minimum degree 3, 4, or 5, and that if the minimum degree is at least 6, then there does not exist any simple counter-example to either conjecture.

Multiple counter-examples to Conjecture 1 can be obtained from simple counter-examples as follows. Take a simple graph H with an irreducible CDC F such that $\chi(H, F) < 2$. For example, H and F can be the graphs in Fig. 2, or H can be any simple 2-connected cubic graph on $n > 4$ vertices and F can be any CDC of H with $|F| \leq n/2$. Let G be the graph obtained from H by adding $m \geq 1$ parallel edges, e_i , $i = 1, \dots, m$, to each edge $e \in E(H)$. Recursively, we construct a sequence $\{F_s\}$ with $F_1 = F$: if there is an edge e contained in two cycles of F_{s-1} , say D is one of them, then $F_s = (F_{s-1} \setminus \{D\}) \cup \{D - e + e_1\}$. Eventually, F will be transformed to a cycle decomposition of $H \cup \{e_1 : e \in E(H)\}$. Now this cycle decomposition can be easily extended to an irreducible CDC C of G by adding appropriate 2-gons (cycles of length 2). Since $|V(G)| = |V(H)|$, $|E(G)| = (m+1)|E(H)|$, and $|C| = |F| + m|E(H)|$, $\chi(G, C) = \chi(H, F) < 2$. Since every cycle of G of length at least three has at least three chords, and since every 2-gon not in C (if any) must have a chord, every cycle not in C is a discrete Jordan curve.

To construct a multiple counter-example of Conjecture 2, we take a simple 2-connected counter-example H with a CDC F of Conjecture 2 such that $\{u, v\}$ is a 2-cut of H with $uv \in E(H)$. For example, H can be taken as the graphs in Fig. 2. Obtain a multigraph G from H by adding $m \geq 1$ multi-

edges between u and v . Extend F to an irreducible CDC C of G by adding appropriate 2-gons. One can easily see that $\chi(G, C) = \chi(H, F) < 2$, and every cycle not in C is separating.

3. CONTRACTIBLE EDGES AND NON-SEPARATING CYCLES

In this section, we introduce *contractible edges* in order to obtain some useful results on non-separating cycles. The graphs discussed in this section are simple. To contract an edge $f = xy$ in a graph G is to identify x and y and to delete all resulting multiedges and loops. The resulting simple graph is denoted by Gf , and the new vertex is denoted by f^* . An edge f in a 3-connected graph G is *contractible* if Gf is also 3-connected. The following three conditions are equivalent for an edge $f = xy$ in a 3-connected graph $G \neq K_4$: (1) f is contractible, (2) $G \setminus \{x, y\}$ is 2-connected, and (3) $\{x, y\}$ is not contained in any 3-cut of G .

There are many results concerning contractible edges. The result that we will use is about covering contractible edges in 3-connected graphs. Let $E_c(G)$ be the set of contractible edges of a 3-connected graph G . We say that $E_c(G)$ is covered by a set $S \subset V(G)$ if each edge of $E_c(G)$ is incident with a vertex of S . The first result in this direction is due to Ando, Enomoto, and Saito [2]; they showed that *if G is a 3-connected graph other than K_4 , then $E_c(G)$ cannot be covered by one vertex*. Later, Ota and Saito [7] came up with the following result.

THEOREM 3.1. *If G is a 3-connected graph other than K_4 , then $E_c(G)$ can be covered by two vertices if and only if G is one of the following two graphs: W_4 and $W_4 + e$. (See Fig. 3).*

Note. The 3-connected graphs whose contractible edges can be covered by exactly three vertices are also characterized by Hemminger and Yu [4] (in this case, an infinite family with a few sporadic graphs).

Let $f = xy$ be a contractible edge of a simple 3-connected graph G with n vertices. Then Gf is a 3-connected graph on $n - 1$ vertices. For an induced cycle γ of Gf we extend it to an induced cycle γ' of G in the following way. If $f^* \notin \gamma$, then let $\gamma' = \gamma$. Suppose that $f^* \in \gamma$. Let u, v be the two vertices of γ adjacent to f^* . If both u and v are adjacent to x (or y), then let $\gamma' = \gamma - f^* + uxv$ (or $\gamma' = \gamma - f^* + uyv$). Otherwise, we may assume that $ux, vy \in E(G)$ and $uy, vx \notin E(G)$, then let $\gamma' = \gamma - f^* + uxyv$. It is easy to check that γ' is an induced cycle.

Since G is 3-connected, if γ is non-separating, then (by checking all the above extensions) γ' is also non-separating unless $\gamma' = \gamma - f^* + uxv$ (or $\gamma' = \gamma - f^* + uyv$) and $N_G(y) = \{x, u, v\}$ (or $N_G(x) = \{y, u, v\}$). In the

exceptional case, we have $d_G(x) \geq 4$ (or $d_G(y) \geq 4$), and so $\gamma - f^* + u y v$ (or $\gamma - f^* + u x v$) is non-separating. Hence, we can always take γ' to be non-separating. On the other hand, if $\gamma'_1 = \gamma'_2$ in G , then $\gamma_1 = \gamma_2$ in Gf . Therefore, distinct induced non-separating cycles of Gf can be extended to distinct induced non-separating cycles of G . Moreover, if S is a set of vertices of G , and $x, y \notin S$, then that γ does not contain S implies that γ' does not contain S . This will be used to take care of the extension parts in the following proofs.

The following theorem which will be used in Section 4 is a direct application of Theorem 3.1.

THEOREM 3.2. *Every simple 3-connected graph has two induced non-separating cycles having an edge in common and each missing at least one of two specified vertices.*

Proof. We use induction on $n = |V(G)|$. The case $n = 4$ is trivial. So let $n \geq 5$. If G is one of the two graphs in Fig. 3, then the claim of the theorem can be easily checked. Hence, by Theorem 3.1, we may assume that G has a contractible edge f not incident to any of the two specified vertices. Now Gf is also simple and 3-connected containing the two specified vertices which are different from f^* . Thus, Gf has two induced non-separating cycles containing a common edge such that each of which misses at least one of the two specified vertices. By the above extension, these two cycles can be easily extended to two desired cycles of G .

Note that the number of cycles in Theorem 3.2 is best possible because of the 3-connected simple planar graphs (see Theorem 2.3). But for non-planar graphs, we can say more.

THEOREM 3.3. *If G is a non-planar simple 3-connected graph, then for any given pair S of vertices there are at least three induced non-separating cycles having an edge in common and each missing at least one vertex of S .*

Proof. We use induction on $n = |V(G)|$. When $n = 5$, $G = K_5$, and the above claim is easy to verify. So let $n \geq 6$. Then, by Theorem 3.1, G has a

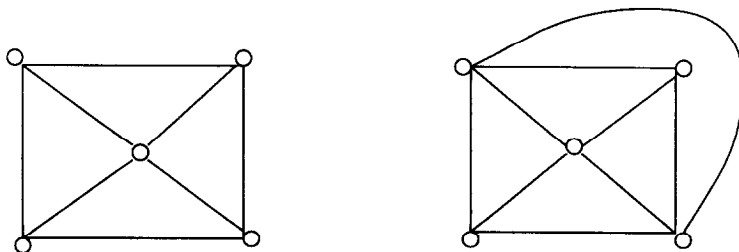


FIGURE 3

contractible edge, say $f=xy$, such that neither x nor y is in S . Consider the 3-connected graph $H=Gf$.

If H is non-planar, then by induction, H has three induced non-separating cycles, say $\gamma_i, i=1, 2, 3$, each missing at least one vertex of S but containing a common edge g of H . By the extensions discussed above, G has three induced non-separating cycles $\gamma'_i, i=1, 2, 3$, each missing at least one vertex of S . Let z be an end of g with $z \neq f^*$. Again, by the above extensions, $g \in \gamma'_i, i=1, 2, 3$, except that $g=zf^*$ and z is adjacent to both x and y in G . In this exceptional case, we have that, for each i , if $zx \in \gamma'_i$, then $zy \notin \gamma'_i$, and vice versa. Without loss of generality, we may assume that $zy \in \gamma'_i, i=1, 2$. Note that, $f=xy \in E_c(G)$, and so, the cycle $xyzx$ is an induced non-separating cycle of G . Therefore, γ'_1, γ'_2 , and $xyzx$ are the three desired cycles.

So suppose that H is a 3-connected planar graph. By Theorem 2.3, all facial cycles of the unique plane embedding of H are the induced non-separating cycles. Let $\gamma = \{v \in H - f^*: v, f^* \text{ are in a common facial cycle of } H\}$. Since H is 3-connected, γ is a cycle in H . Obviously, γ is also a cycle in G , and every neighbor of x or y other than themselves is in γ . Let y be adjacent to $u_i, i=1, \dots, s$, on γ in a clockwise order. Clearly, $s \geq 2$. Let P_i be the path on γ from u_i to $u_{i+1}, i=1, \dots, s$, where the subscripts are modulo s . Since G is not planar, there are integers k and $l, 1 \leq k < l \leq s$, such that x is adjacent to some vertex of $P_k \setminus P_l$ as well as to some vertex of $P_l \setminus P_k$. Note that $k=l+1$ or $l=k+1$ is possible. In the clockwise order,

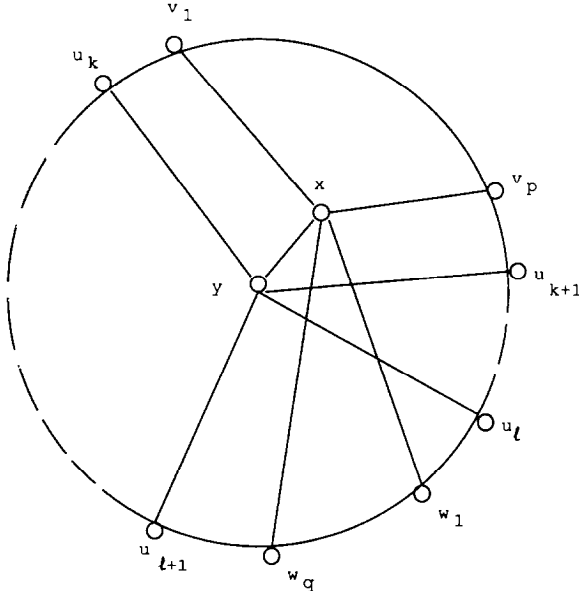


FIGURE 4

let x be adjacent to v_r , $r = 1, \dots, p$, on P_k , and to w_t , $t = 1, \dots, q$, on P_l . See Fig. 4.

It is clear that if $v_1 \neq u_{k+1}$, then v_1xyu_k together with the path on γ from u_k to v_1 is an induced non-separating cycle of G containing the edge xy ($v_1 = u_k$ is possible). The same argument will produce another three induced non-separating cycles of G containing xy if $v_p \neq u_k$, $w_1 \neq u_{l+1}$, and $w_q \neq u_l$. Obviously, at least three of the four induced non-separating cycles do not contain S (since neither x nor y is in S). Thus, $f = xy$ is contained in at least three induced non-separating cycles each missing at least one vertex of S . Hence, we may assume, without loss of generality, that $v_p = u_k$, and so $p = 1$. Thus, $u_{l+1} \neq u_k$ (since $v_p \notin P_l$).

We claim that we may assume $S \subset P_{k-1}$ or $S \subset P_k$. First, we may assume that $N_G(x) \cap (P_{k-1} \setminus \{u_{k-1}, u_k\}) \neq \emptyset$. Otherwise, $yP_{k-1}y$, yP_ky , and yu_kxy are three induced non-separating cycles of G containing the edge yu_k . These cycles are the desired cycles except $S \subset P_{k-1}$ or $S \subset P_k$, and so the claim follows. Hence, let $z \in N_G(x) \cap (P_{k-1} \setminus \{u_{k-1}, u_k\})$ such that z is closest to u_{k-1} (on P_{k-1}). Let Q be the subpath of P_{k-1} from u_{k-1} to z . By the argument of the last paragraph (with P_k being replaced by P_{k-1}), we have either $w_1 = u_{l+1}$ or $w_q = u_l$. Hence, there are three non-separating cycles containing the edge xy : yu_kxy , $yQxy$, and yw_1xy . These three cycles are the desired cycles except when $S \subset Q$ or when $yQxy$ is not induced. When $yQxy$ is not induced, $w_1 = u_{l+1} = u_{k-1}$. Since G is not planar, there is some $j \neq k, l$ such that x is adjacent to some vertex of $P_j \setminus P_{k-1}$. Once again, we have that x is adjacent to u_j or u_{j+1} . Hence, there are three triangles containing the edge xy , and these three triangles are the desired cycles. Thus, we may assume $S \subset Q$, and so the claim follows.

We may also assume $w_1 \in \{u_l, u_{l+1}\}$. Otherwise, let Q_1 be the subpath of P_l from w_q to u_{l+1} , and Q_2 be the subpath of P_l from u_l to w_1 . Then yu_kxy , yQ_1xy , and yQ_2xy are three desired cycles (since $S \subset P_{k-1}$ or $S \subset P_k$).

If degree of x is not three, then there is another P_j , $j \neq k$ and $j \neq l$, such that x is adjacent to a vertex of P_j other than v_1 or w_1 . Then, the above argument allows us to assume that z is adjacent to a unique vertex of P_j and it is u_j or u_{j+1} . Thus, the edge xy is contained in three triangles each of which is induced and non-separating (since $f = xy \in E_c(G)$). Since $x, y \notin S$, each of these three triangles misses at least one vertex of S . Hence, we may assume that x has degree three in G .

If $w_1 = u_l$, then $k+1 \neq l$, otherwise G would be planar. Since $S \subset P_{k-1} \cup P_k$, the three cycles $yP_{l-1}y$, yP_ly and yxu_ly are the desired cycles (containing a common edge yu_l). So $w_1 = u_{l+1}$. Then $k \neq l+1$ and $k \neq l+2$, otherwise G would be planar. Again, since $S \subset P_{k-1} \cup P_k$, the cycles yP_ly , $yP_{l+1}y$ and $yu_{l+1}xy$ are the desired cycles (containing a common edge yu_{l+1}). This completes our proof.

COROLLARY 3.4. *If G is a simple 3-connected non-planar graph, then G has at least two induced non-separating cycles outside a given CDC.*

Proof. Since G is non-planar, by Theorem 2.3, G has an edge e contained in at least three induced non-separating cycles of G . By the above theorem, G has at least three induced non-separating cycles containing a common edge and each missing at least one end of e . Therefore, all these six cycles are distinct. Any CDC of G can contain at most four of these cycles, and so, the corollary follows.

Theorem 3.3 is best possible by the graphs K_5 and $K_{3,3}$. The following results are also corollaries of Theorem 3.3. But first we need the following notation. Let S be a 2-cut of a graph G and let B be a connected component of $G \setminus S$. Then $B^+ = G[B \cup S]$ if $G[S]$ is an edge, otherwise $B^+ = G[B \cup S] + e$ where e is an edge joining the two vertices of S . A component B is *minimal* if B^+ contains no 2-cut. Thus, if B is minimal, then B^+ is either a 3-connected graph or a triangle.

THEOREM 3.5. *If G is a 2-connected simple graph with a minimal component B with respect to a 2-cut S such that B^+ is non-planar, then B^+ has an induced non-separating cycle of G outside a given CDC of G .*

Proof. Since B^+ is non-planar, B^+ is a 3-connected graph. By Theorem 3.2, B^+ has an edge which is contained in at least three induced non-separating cycles γ_i , $i = 1, 2, 3$, not containing S . Therefore, each γ_i is contained in B^+ and is an induced non-separating cycle of G . Clearly, at least one of these three cycles is not in a given CDC of G .

THEOREM 3.6. *If G is a simple 2-connected graph with minimum degree at least 6, then G has at least two induced non-separating cycles outside a given CDC.*

Proof. We can assume that G is not 3-connected by Corollary 3.4. Take any minimal component B with respect to a 2-cut S of G . Since the minimum degree of G is at least six, B^+ is 3-connected. Note that every vertex in B^+ has degree at least six except that those two in S have degree at least three. Hence, $|E(B^+)| \geq (6(|B^+| - 2) + 3 + 3)/2 = 3|B^+| - 3$, and so, B^+ is non-planar. By Theorem 3.5, for any given CDC C , B^+ has an induced non-separating cycle of G outside C . Now, the theorem follows from the fact that G has at least two minimal components with respect to 2-cuts of G .

COROLLARY 3.7. *Conjectures 1 and 2 are true for simple 2-connected graphs with minimum degree at least six (even without the conditions that $\chi(G, C) \neq 2$ and that C is irreducible).*

The graphs of Fig. 2 are of minimum degree 2. To see that the minimum degree condition of Corollary 3.7 cannot be relaxed, we construct simple 2-connected counterexamples to Conjectures 1 and 2 with minimum degree equal to any of 3, 4, and 5.

As in the end of Section 2, let H be a 2-connected graph with an irreducible CDC F such that $\chi(H, F) < 2$. For each edge $e \in E(H)$, let H_e be a 3-connected planar graph with minimum degree r ($3 \leq r \leq 5$). Let F_e be the CDC of H_e consisting of all facial cycles of H_e . Select an edge $f(e)$ of H_e so that $H_e - f(e)$ also has a vertex of degree r . Now construct a new simple 2-connected graph G from H by replacing each edge $e \in E(H)$ by $H_e - f(e)$ with the ends of e and $f(e)$ identified. Then G has minimum degree r . Let $P_1(e)$ and $P_2(e)$ be the two paths of H_e such that $P_1(e) + f(e)$ and $P_2(e) + f(e)$ are the two facial cycles of H_e containing $f(e)$. We construct a CDC C of G from F as follows: for each $e \in E(G)$, all facial cycles of H_e not containing $f(e)$ are in C ; if $\gamma \in F$, then replace each $e \in \gamma$ by $P_1(e)$ if the edges of $P_1(e)$ are used only once (as parts of facial cycles of H_e), otherwise replace e by $P_2(e)$.

Obviously, C is a CDC of G , and it is not difficult to show that C is irreducible (otherwise, F would be reducible). Now, $|V(G)| = |V(H)| + \sum_{e \in E(H)} (|V(H_e)| - 2)$, $|E(G)| = \sum_{e \in E(H)} (|E(H_e)| - 1)$, and $|C| = |F| + \sum_{e \in E(H)} (|F_e| - 2)$. Hence, $\chi(G, C) = \chi(H, F) < 2$ (since $\chi(H_e, F_e) = 2$ for each $e \in E(H)$). Clearly, every cycle of G not in C is a separating cycle (hence a discrete Jordan curve). Note that G is planar if and only if H is planar.

4. GRAPHS WITH NO INDUCED NON-SEPARATING CYCLES

Let $\mathbf{G} = \{G : G \text{ is a simple 2-connected graph in which every cycle is separating}\}$. A k -rail in a graph G between two vertices is a set of k internally vertex disjoint paths such that all internal vertices are of degree 2 in G . The following result due to Thomassen and Toft [10] is an easy corollary of Theorem 3.2.

THEOREM 4.1. *Let $G \in \mathbf{G}$. Then there exists a pair of vertices $\{x, y\}$ such that there is a 3-rail between x and y in G . Moreover, either G is a k -rail for some $k \geq 4$ or there exists another pair of vertices $\{x', y'\}$ (distinct but not necessarily disjoint from $\{x, y\}$) such that there is a 3-rail between x' and y' .*

Proof. Suppose that the above claim is not true. Select a counter-example G so that $|V(G)|$ is minimum. Then, clearly, G has no edge joining two degree 2 vertices (otherwise, identify the two degree two vertices and delete the resulting loop, we would have a smaller counter-example). Also, G

cannot be 3-connected (by Theorem 3.2). So let B be a minimal component of $G \setminus S$ for some 2-cut $S = \{u, v\}$. Then, by Theorem 3.2, $|B| = 1$ (otherwise, B^+ would be 3-connected). Let $w \in B$. Now construct a graph H from G as follows: $H = G - w + uv$ if $uv \notin E(G)$, or $H = G - w$ if $uv \in E(G)$. In each case, it is easy to show that $H \in \mathbf{G}$ and that both u and v have degree at least three in H . Hence, any k -rail of H is also a k -rail of G for $k \geq 3$. By the choice of G , H satisfies the claim of the above theorem, and so does G , a contradiction which completes the proof.

A corollary of the above theorem is that any 2-connected graph with minimum degree at least 3 has an induced non-separating cycle. By using Theorem 3.2, we have the following stronger result.

THEOREM 4.2. *Let G be a simple 2-connected graph with minimum degree 3. Then G has at least two induced non-separating cycles.*

Proof. Obviously, G must have a 2-cut. Thus, G must have two minimal components with respect to some 2-cuts of G . By Theorem 3.2, if G has at most one induced non-separating cycle, then at least one minimal component is a single vertex, and so, G has a vertex of degree 2, a contradiction.

Finally, we study a conjecture of Thomassen and Toft [10] about the structure of graphs in \mathbf{G} . Let $G \in \mathbf{G}$. Let $v_2(G)$ denote the number of degree 2 vertices in G , and $s_2(G)$ the number of 2-cuts of G . Thomassen and Toft [10] made the following conjecture:

Conjecture 3. There is a positive constant c such that $v_2(G) + s_2(G) > c|E(G)|$ for any $G \in \mathbf{G}$.

We now disprove this conjecture.

Let K_n be the complete graph on $n \geq 3$ vertices. For each positive integer $m \geq 2$, construct the graph G_m from K_n as follows: (1) add m new vertices and join each new vertex to all vertices of K_n , and (2) for each edge uv of K_n , add two more new vertices and join each of these two vertices to both u and v .

Clearly, every induced cycle of G_m is a triangle. But every triangle of G_m contains a 2-cut. Hence, $G_m \in \mathbf{G}$. Now, $v_2(G_m) = n(n-1)$, $s_2(G_m) = n(n-1)/2$, and $|E(G_m)| = nm + 5n(n-1)/2$. Hence, $(v_2(G_m) + s_2(G_m))/|E(G_m)| \rightarrow 0$ as $m \rightarrow \infty$; that is, such a constant in Conjecture 3 does not exist.

Note that the graphs that we constructed above are non-planar and have girth three. For planar graphs, one can easily show that such a constant c exists which raises the question: What is the best such number?

For graphs with girth four, we can also show that Conjecture 3 is not

true by constructing a graph H_m from the complete bipartite graph $K_{m,n}$ as follows: for any two distinct vertices u, v in the partition of $K_{m,n}$ with cardinality n , add three new vertices and join each of them to both u and v . Clearly, H_m has girth 4 and every cycle of H_m contains a 2-cut (and so $H_m \in \mathbf{G}$). An easy calculation shows that $(v_2(H_m) + s_2(H_m))/|E(H_m)| \rightarrow 0$ as $m \rightarrow \infty$; that is, Conjecture 3 is not true for graphs with girth 4.

This leads to the next question: Is there a positive integer $g \geq 5$ such that Conjecture 3 is true for graphs in \mathbf{G} with girth at least g ?

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